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COLLAPSING RIEMANNIAN MANIFOLDS TO ONES OF LOWER DIMENSIONS

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0. Introduction

In [7], Gromov introduced a notion, Hausdorff distance, between two metric spaces. Several authors found that interesting phenomena occur when a sequence of Riemannian manifolds M_i collapses to a lower dimensional space X. (Examples of such phenomena will be given later.) But, in general, it seems very difficult to describe the relation between topological structures of M_i and X. In this paper, we shall study the case when the limit space X is a Riemannian manifold and the sectional curvatures of M_i are bounded, and shall prove that, in that case, M_i is a fiber bundle over X and the fiber is an infranilmanifold. Here a manifold F is said to be an infranilmanifold if a finite covering of F is diffeomorphic to a quotient of a nilpotent Lie group by its lattice.

A complete Riemannian manifold M is contained in class $\mathscr{M}(n)$ if **dim** $M \leqslant n$ and if the sectional curvature of M is smaller than 1 and greater than -1. An element N of $\mathscr{M}(n)$ is contained in $\mathscr{M}(n, \mu)$ if the injectivity radius of N is everywhere greater than μ.

Main Theorem. *There exists a positive number $\varepsilon(n, \mu)$ depending only on n and μ such that the following holds.*

If $M \in \mathscr{M}(n)$, $N \in \mathscr{M}(n, \mu)$, and if the Hausdorff distance ε between them is smaller than $\varepsilon(n, \mu)$, then there exists a map $f: M \to N$ satisfying the conditions below.

(0-1-1) *(M, N, f) is a fiber bundle.*

(0-1-2) *The fiber of f is diffeomorphic to an infranilmanifold.*

(0-1-3) *If $\xi \in T(M)$ is perpendicular to a fiber of f, then we have*

$$e^{-\tau(\varepsilon)} < |df(\xi)|/|\xi| < e^{\tau(\varepsilon)}.$$

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Here $\tau(\varepsilon)$ is a positive number depending only on ε, n, μ and satisfying $\lim_{\varepsilon\to 0}\tau(\varepsilon) = 0$.

Remarks. (1) In the case when N is equal to a point, our main theorem coincides with [6, 1.4].

(2) In the case when the dimension of M is equal to that of N, the conclusion of our main theorem implies that f is a diffeomorphism and that the Lipschitz constants of f and f^{-1} are close to 1. Hence, in that case, our main theorem gives a slightly stronger version of [7, 8.25] or [8, Theorem 1]. (In [7] or [8], it was assumed that the injectivity radii of both M and N were greater than μ, but here we assume that one of them is greater than μ.)

Next we shall give some examples illustrating the phenomena treated in our main theorem.

Examples. (1) Let $T_i^2 = \mathbb{R}^2/\mathbb{Z} \oplus (1/i)\mathbb{Z}$ be flat tori. Then $\lim_{i\to\infty} T_i^2 = S^1$ $(= \mathbb{R}/\mathbb{Z})$ and T^2 is a fiber bundle over S^1.

(2) (See [9].) Let (M, g) be a Riemannian manifold. Suppose S^1 acts isometrically and freely on M. Let g_ε denote the Riemannian metric such that $g_\varepsilon(v, v) = \varepsilon \cdot g(v, v)$ if v is tangent to an orbit of S^1 and $g_\varepsilon(v, v) = g(v, v)$ if v is perpendicular to an orbit of S^1. Then $\lim_{\varepsilon\to 0}(M, g_\varepsilon) = (M/S^1, g')$ for some metric g'. In this example, the fiber bundle in our main theorem is $S^1 \to M \to M/S^1$.

(3) Let G be a solvable Lie group and Γ its lattice. Put $G_0 = G$, $G_1 = [G, G]$, $G_2 = [G_1, G_1], \cdots, G_{i+1} = [G_1, G_i]$. Take a left invariant Riemannian metric g on G. Let g_ε denote the left invariant Riemannian metric on G such that $g_\varepsilon(v, v) = \varepsilon^{i\cdot 2^i} \cdot g(v, v)$ if $v \in T_e(G)$ is tangent to G_i and perpendicular to G_{i+1}. (Here e denotes the unit element.) Then $\lim_{\varepsilon\to 0}(\Gamma\setminus G, g_\varepsilon)$ is equal to the flat torus $\Gamma\setminus G/G_1$, and the sectional curvatures of g_ε are uniformly bounded. In this example, the fiber bundle in our main theorem is $(G_1 \cap \Gamma)\setminus G_1 \to \Gamma\setminus G \to \Gamma\setminus G/G_1$.

Finally, we shall give an example of collapsing to a space which is not a Riemannian manifold.

(4) (This example is an amplification of [7, 8.31].) Let (G_i, Γ_i) be a sequence of pairs consisting of nilpotent Lie groups G_i and their lattices Γ_i. Let (M, g) be a compact Riemannian manifold and φ_i a homomorphism from Γ_i to the group of isometries of (M, g). Put $T = \bigcap_i(\overline{\bigcup_{j\geq i}\varphi_j(\Gamma_j)})$. Here the closure, $\overline{\bigcup_{j\geq i}\varphi_j(\Gamma_j)}$, is taken in the sense of compact open topology. It is proved in [1, 7.7.2] that there exists a sequence of left invariant metrics g_i on G_i such that the sectional curvatures of g_i $(i = 1, 2, \cdots)$ are uniformly bounded and that $\lim_{i\to\infty}(\Gamma_i\setminus G_i, \bar{g}_i) =$ point. On $M \times G_i$, we define an equivalence relation $\sim$ by $(\varphi_i(\gamma^{-1})(x), g) \sim (x, \gamma g)$. Let $M \times_{\Gamma_i} G_i$ denote the set of equivalence

classes. Then it is easy to see

$$\lim_{i\to\infty}\left(M\times_{\Gamma_i}G_i,\ \overline{g\times g_i}\right)=(M/T,\bar{g}).$$

In this example, there also exists a map from $M\times_{\Gamma_i}G_i$ to M/T.

This example gives all possible phenomena which can occur at a neighborhood of each point of the limit. In fact, using the result of this paper, we shall prove the following in [5]:

Let M_i be a sequence of compact m-dimensional Riemannian manifolds such that the sectional curvatures of M_i are greater than -1 and smaller than 1. Suppose $\lim_{i\to\infty}M_i$ is equal to a compact metric space X. Then, for each sufficiently large i, there exists a map $f\colon M_i\to X$ satisfying the following.

(1) For each point p of X, there exists a neighborhood U which is homeomorphic to the quotient of $\mathbb{R}^n$ by a linear action of a group T. Here T denotes an extension of a torus by a finite group.

(2) Let Y denote the subset of X consisting of all points which have neighborhoods homeomorphic to $\mathbb{R}^k$. Then $(f_i|_{f_i^{-1}(Y)}, f_i^{-1}(Y), Y)$ is a fiber bundle with an infranilmanifold fiber F.

(3) Suppose p has a neighborhood homeomorphic to $\mathbb{R}^n/T$. Then $f_i^{-1}(p)$ is diffeomorphic to F/T.

The global problem on collapsing is still open even in the case of fiber bundles.

Problem. Let F be an infranilmanifold and (M, N, f) a fiber bundle with fiber F. Give a necessary and sufficient condition for the existence of a sequence of metrics g_i on M such that the sectional curvatures are greater than -1 and smaller than 1 and that $\lim_{i\to\infty}(M, g_i)$ is homeomorphic to N.

The organization of this paper is as follows. In §1, we shall construct the map f. In §2, we shall prove that (M, N, f) is a fiber bundle. In §3, we shall prove a lemma on triangles on M. This lemma will be used in the argument of §§2, 4, and 5. In §4, we shall verify (0-1-3). In §5, we shall prove (0-1-2). Our argument there is an extension of one in [1] or [6].

In [7, Chapter 8] and [9] (especially in [7, 8.52]), several results which are closely related to this paper are proved or announced, and the author is much inspired from them. Several related results are obtained independently in [3] and [4]. The result of this paper is also closely related to Thurston's proof of his theorem on the existence of geometric structures on 3-dimensional orbifolds. The lecture by T. Soma on it was also very helpful to the author.

Notation. Put $R=\min(\mu,\pi)/2$. The symbol ε denotes the Hausdorff distance between M and N. Let σ be a small positive number which does not depend on ε. We shall replace the numbers ε and σ by smaller ones, several times in the proof. The symbol $\tau(a\,|\,b,\cdots,c)$ denotes a positive number depending only on $a, b,\cdots, c, R, \mu$ and satisfying $\mathbf{lim}_{a\to 0}\tau(a\,|\,b,\cdots,c) = 0$ for each fixed $b,\cdots,c$. For a Riemannian manifold X, a point $p \in X$, and a positive number r, we put

$$B_r(p, X) = \{x \in X\,|\,d(x, p) < r\},$$
$$BT_r(p, X) = \{\xi \in T_p(X)\,|\,|\xi| < r\}.$$

Here $T_p(X)$ denotes the tangent space. For a curve $l\colon[0,T] \to X$, we let $(Dl/dt)(t)$ denote the tangent vector of l at $l(t)$. For two vectors $\xi, \xi' \in T_p(X)$, we let $\mathrm{ang}(\xi, \xi')$ denote the angle between them. All geodesics are assumed to have unit speed.

1. Construction of the map

First remark that Rauch's comparison theorem (see [2, Chapter 1, §1]) immediately implies the following.

(1-1-1) For each $p \in M$ and $p' \in N$ the maps $\mathbf{exp}|_{BT_{2R}(p,M)}$ and $\mathbf{exp}|_{BT_{2R}(p',N)}$ have maximal rank. Here **exp** denotes the exponential map.

(1-1-2) On $BT_{2R}(p, M)$ [resp. $BT_{2R}(p', N)$], we define a Riemannian metric induced from M [resp. N] by the exponential map. Then, the injectivity radii are greater than R on $BT_R(p, M)$ and $BT_R(p', N)$.

Secondly we see that, by the definition of the Hausdorff distance, there exists a metric d on the disjoint union of M and N such that the following holds: The restrictions of d to M and N coincide with the original metrics on M and N respectively, and for each $x \in N$, $y \in M$ there exist $x' \in M$, $y' \in N$ such that $d(x, x') < \varepsilon$, $d(y, y') < \varepsilon$. It follows that we can take subsets Z_N of N and Z_M of M, a set Z, and bijections $j_M\colon Z \to Z_M$, $j_N\colon Z \to Z_N$, such that the following holds.

(1-2-1) The 3ε-neighborhood of Z_N [resp. Z_M] contains N [resp. M].

(1-2-2) If z and z' are two elements of Z, then we have

$$d(j_N(z), j_N(z')) > \varepsilon \quad \text{and} \quad d(j_M(z), j_M(z')) > \varepsilon.$$

(1-2-3) For each $z \in Z$, we have

$$d(j_N(z), j_M(z)) < \varepsilon.$$

Now, following [8], we shall construct an embedding $f_N\colon N \to \mathbb{R}^Z$. Put $r = \sigma R/2$. Let κ be a positive number determined later, and $h\colon\mathbb{R} \to [0,1]$ a

C^∞-function such that

(1-3) $h(0) = 1$ and $h(t) = 0$ if $t \geqslant r$,

$$\tfrac{4}{r} < h'(t) < -\tfrac{3}{r} \quad \text{if } \tfrac{3r}{8} < t < \tfrac{5r}{8},$$

$$-\tfrac{4}{r} < h'(t) < 0 \quad \text{if } \tfrac{2r}{8} < t \leqslant \tfrac{3r}{8} \text{ or } \tfrac{5r}{8} \leqslant t < \tfrac{6r}{8},$$

$$\kappa < h'(t) < 0 \quad \text{if } 0 < t < \tfrac{2r}{8} \text{ or } \tfrac{6r}{8} \leqslant t \leqslant r.$$

We define a C^∞-map $f_N : N \to \mathbb{R}^Z$ by $f_N(x) = (h(d(x, j_N(z))))_{z \in Z_N}$. In [8], it is proved that, if ε and σ are smaller than a constant depending only on R, μ, and n, then f_N satisfies the following facts (1-4-1), (1-4-2), (1-4-3), and (1-4-4). The numbers C_1, C_2, C_3, C_4 below are positive constants depending only on R, μ, and n.

(1-4-1) f_N is an embedding [8, Lemma 2.2].

(1-4-2) Put

$$B_C(Nf_N(N)) = \{(p, u) \in \text{the normal bundle of } f_N(N) \mid |u| < C\},$$

$$K = \sup_{x \in N} \#(B_r(p, N) \cap j_N(Z_N)).$$

Then the restriction of the exponential map to $B_{C_1 K^{1/2}}(Nf_N(N))$ is a diffeomorphism [8, Lemma 4.3].

(1-4-3) For each $\xi' \in T_{p'}(N)$ satisfying $|\xi'| = 1$, we have

$$C_2 K^{1/2} < |df_N(\xi')| < C_3 K^{1/2} \quad [8, \text{Lemma } 3.2].$$

(1-4-4) Let $x, y \in N$. If $d(x, y)$ is smaller than a constant depending only on σ, μ, and n, then we have

$$K^{1/2} \cdot d(x, y) \leqslant C_4 \cdot d_{\mathbb{R}^n}(f_N(x), f_N(y)) \quad [8, \text{Lemma } 6.1].$$

The next step is to construct a map from M to the $C_1 K^{1/2}$-neighborhood of $f_N(N)$. The map $x \to (h(d(x, j_M(z))))_{z \in Z}$ has this property. But unfortunately this map is not differentiable when the injectivity radius of M is smaller than r, and is inconvenient for our purpose. Hence we shall modify this map. For $z \in Z$ and $x \in M$, put

$$d_z(x) = \int_{y \in B_\varepsilon(j_M(z), M)} d(y, x)\, dy / \mathrm{Vol}(B_\varepsilon(j_M(z), M)),$$

$$f_M(x) = (h(d_z(x)))_{(z \in Z)}.$$

Assertion 1-5. *d_z is a C^1-function and for each $\xi \in T_x(M)$ we have*

$$\xi(d_z) = \frac{\int_A \xi(d(y, \cdot))\, dy}{\mathrm{Vol}(A)}.$$

Here $A = \{ y \in B_\varepsilon(j_M(z), N) \mid y$ is not a cut point of $x \}$.

Assertion 1-5 is a direct consequence of the following two facts: d_z *is a Lipschitz function*; *the cut locus is contained in a set of smaller dimension.* (*Remark that d_z is not necessarily of C^2-class.*)

Lemma 1-6. *$f_M(M)$ is contained in the $3\varepsilon K^{1/2}$-neighborhood of $f_N(N)$.*

Proof. Let x be an arbitrary point of M. The definition of d_z implies $|d(j_M(z), x) - d_z(x)| < \varepsilon$. Take a point x' of N such that $d(x, x') < \varepsilon$. Then condition (1-2-3) implies that $|d(j_M(z), x) - d(j_N(z), x')| < 2\varepsilon$. It follows that $|d(j_N(z), x') - d_z(x)| < 3\varepsilon$. The lemma follows immediately.

Lemma 1-6, combined with facts (1-4-1) and (1-4-2), implies that $f_N^{-1} \circ \pi \circ \mathrm{Exp}^{-1} \circ f_M = f$ is well defined, where $\pi : N(f_N(N)) \to f_N(N)$ denotes the projection. This is the map f in our main theorem.

2. (M, N, f) is a fiber bundle

The proof of the following lemma will be given in the next section. Let δ, δ', and ν be positive numbers satisfying $\delta \leqslant \delta'$.

Lemma 2-1. *Let $l_i : [0, t_i] \to M (i = 1, 2)$ be geodesics on M such that $l_1(0) = l_2(0)$, and $l_i' : [0, t_i'] \; (i = 1, 2)$ be minimal geodesics on N such that $l_1'(0) = l_2'(0)$. Suppose*

(2-2-1) $$d(l_i(0), l_i(t_i)) - t_i < \nu,$$

(2-2-2) $$d(l_i(0), l_i'(0)) < \nu,$$

(2-2-3) $$d(l_i(t_i), l_i'(t_i')) < \nu,$$

(2-2-4) $$\delta R/10 < t_1 < \delta R \quad \textit{and} \quad \delta' R/10 < t_2 < \delta' R.$$

Then we have

$$\left| \operatorname{ang}\left(\frac{Dl_1}{dt}(0), \frac{Dl_2}{dt}(0) \right) - \operatorname{ang}\left(\frac{Dl_1'}{dt}(0), \frac{Dl_2'}{dt}(0) \right) \right| < \tau(\delta) + \tau(\nu \mid \delta, \delta') + \tau(\varepsilon \mid \delta, \delta').$$

Now we shall show that (M, N, f) is a fiber bundle. It suffices to see that f_M is transversal to the fibers of the normal bundle of $f_N(N)$. (Here we identified the tubular neighborhood to the normal bundle.) For this purpose, we have only to show the following lemma.

Lemma 2-3. *For each $p \in M$ and $\xi' \in T_{f(p)}(N)$, there exists $\xi \in T_p(M)$ satisfying*

$$|df_M(\xi) - df_N(\xi')| / |df_N(\xi')| < \tau(\sigma) + \tau(\varepsilon \mid \sigma).$$

To prove Lemma 2-3, we need Lemmas 2-4 and 2-9.

Lemma 2-4. *Suppose $\sigma \leqslant \delta$, $\nu < \sigma/100$. Let $l_3:[0,t_3] \to M$, $l_3':[0,t_3'] \to N$ be minimal geodesics satisfying the following*

(2-5-1) $$d(l_3(0), l_3'(0)) < \nu,$$

(2-5-2) $$d(l_3(t_3), l_3'(t_3')) < \nu,$$

(2-5-3) $$\delta R/10 < t_3, t_3' < \delta R.$$

Then we have

$$\frac{\left| df_M\left(\frac{Dl_3}{dt}(0)\right) - df_N\left(\frac{Dl_3'}{dt}(0)\right)\right|}{\left| df_N\left(\frac{Dl_3'}{dt}(0)\right)\right|} < \tau(\sigma) + \tau(\nu\,|\,\sigma,\delta) + \tau(\varepsilon\,|\,\sigma,\delta).$$

Proof. Put $p = l_3(0)$, $\xi = (Dl_3/dt)(0)$, $\xi' = (Dl_3'/dt)(0)$. For an arbitrary element z of Z satisfying

(2-6) $$d(p, j_M(z)) > r + 2\nu \quad \text{or} \quad d(p, j_M(z)) < r/8 - 2\nu,$$

we have, by (1.3), that

(2-7) $$|\xi(h(d(j_N(z),\cdot)))| < \kappa, \quad |\xi(h(\tilde{d}_x(\cdot)))| < \kappa,$$

in some neighborhoods of $l_3'(0)$ and $l_3(0)$, respectively. Next we shall study the case when $z \in Z$ does not satisfy (2-6). Assume that an element y of $B_\varepsilon(j_M(z), M)$ is not contained in the cut locus of p. Let $l_4:[0,t_4] \to M$ and $l_4':[0,t_4'] \to N$ denote minimal geodesics joining $l_3(0)$ to y and $l_3'(0)$ to $j_N(z)$ respectively. Since $\sigma R/10 < r/8 - 2\varepsilon - 2\nu < r + 2\varepsilon + 2\nu < \sigma R$, we have $\sigma R/10 < t_4 < \sigma R$, $\delta R/10 < t_3 < \delta R$. Hence, Lemma 2-1 implies

$$|\xi'(d(j_N(z),\cdot)) - \xi(d(y,\cdot))| < \tau(\sigma) + \tau(\nu\,|\,\sigma,\delta) + \tau(\varepsilon\,|\,\sigma,\delta).$$

Therefore, by using Assertion 1-5, we have

(2-8) $$|\xi'(d(j_N(z),\cdot)) - \xi(d_z(\cdot))| < \tau(\sigma) + \tau(\nu\,|\,\sigma,\delta) + \tau(\varepsilon\,|\,\sigma,\delta).$$

Then, Lemma 2-4 follows from (2-7), (2-8), and (1-4-3) if we take κ sufficiently small.

Lemma 2-9. *For each $p \in M$, we have $d(p, f(p)) < \tau(\varepsilon)$.*

Proof. By the definition of f and Lemma 1-6, we have

(2-10) $$d_{\mathbb{R}^n}(f_M(p), f_N(f(p))) < 3\varepsilon K^{1/2}.$$

Let $q \in N$ be an element satisfying $d(p,q) < \varepsilon$. Then, by the proof of Lemma 1-6, we have

(2-11) $$d_{\mathbb{R}^n}(f_M(p), f_N(q)) < 3\varepsilon K^{1/2}.$$

Inequalities (2-10) and (2-11) imply

$$d_{\mathbb{R}^n}(f_N(q), f_N(f(p))) < 6\varepsilon K^{1/2}.$$

Therefore (1-4-4) implies

$$d(q, f(p)) < 6C_4\varepsilon.$$

The above inequality, combined with $d(p, q) < \varepsilon$, implies the lemma.

Proof of Lemma 2-3. By assumption, there exist geodesics $l_3: [0, t_3] \to M$, $l_3': [0, t_3'] \to N$ such that $l_3(0) = p$, $l_3'(0) = f(p)$, $d(l_3(t_3), l_3'(t_3')) < \varepsilon$, $(Dl_3'/dt)(0) = \xi'$, and $\sigma R/10 < t_3, t_3' < \sigma R$. Lemma 2-9 implies $d(l_3(0), l_3'(0)) < \tau(\varepsilon)$. Therefore, Lemma 2-4 implies

$$\left| df_N(\xi') - df_M\left(\frac{Dl_3}{dt}(0)\right)\right| \Big/ |df_N(\xi')| < \tau(\sigma) + \tau(\varepsilon|\sigma),$$

as required.

3. A triangle comparison lemma

To prove Lemma 2-1, we need the following:

Lemma 3-1. *Let* $l_i: [0, t_i] \to M$ $(i = 5, 6)$ *be geodesics on* M *such that* $l_5(0) = l_6(0)$. *Suppose*

(3-2-1) $$l_5(0) = l_5(t_5),$$

(3-2-2) $$|d(l_6(0), l_6(t_6)) - t_6| < \nu,$$

(3-2-3) $$\delta^2 R < t_5 < 2\delta R \quad \textit{and} \quad \delta R/10 < t_6 < \delta R.$$

Then we have

$$\left|\operatorname{ang}\left(\frac{Dl_5}{dt}(0), \frac{Dl_6}{dt}(0)\right) - \pi/2\right| < \tau(\delta) + \tau(\nu|\delta) + \tau(\varepsilon|\delta).$$

Proof. Let $l_6': [-t_6/\delta, t_6/\delta] \to N$ be a minimal geodesic satisfying $d(l_6(0), l_6'(0)) < \varepsilon$ and $d(l_6(t_6), l_6'(t_6)) < 3\varepsilon + \nu$. (The existence of such a geodesic follows from (3-2-2).) Take a minimal geodesic $l_7: [0, t_7] \to M$ satisfying $l_7(0) = l_5(0)$ and $d(l_7(t_7), l_6'(t_6/\delta)) < \varepsilon$. Let $l_8: [0, t_8] \to M$ be a minimal geodesic joining $l_6(t_6)$ to $l_7(t_7)$. Then, since $|t_6 + t_8 - t_7| < \tau(\nu) + \tau(\varepsilon)$, and since l_7 is minimal, it follows that

(3-3) $$\operatorname{ang}\left(\frac{Dl_6}{dt}(t_6), \frac{Dl_8}{dt}(0)\right) < \tau(\nu|\delta) + \tau(\varepsilon|\delta).$$

Let $l_9: [0, t_6/\delta] \to M$ denote the geodesic such that $l_9|_{[0, t_6]} = l_6$. Put $t_9 = t_6/\delta$ $(< R)$. Inequality (3-3) and the fact $|t_7 - t_9| < \tau(\nu) + \tau(\varepsilon)$ imply

$$d(l_7(t_7), l_9(t_9)) < \tau(\nu|\delta) + \tau(\varepsilon|\delta).$$

Hence, by the minimality of l_7, we obtain

$$|d(0, l_9(t_9)) - t_9| < \tau(\nu|\delta) + \tau(\varepsilon|\delta). \tag{3-4}$$

Now let $\tilde{l}_i : [0, t_i] \to BT_R(l_1(0), M)$ $(i = 5, 9)$ denote the lifts of l_i such that $\tilde{l}_i(0) = 0$. Then, (3-4) implies

$$d(\tilde{l}_5(t_5), \tilde{l}_9(t_9)) > d(\tilde{l}_5(0), \tilde{l}_9(t_9)) - \tau(\nu|\delta) - \tau(\varepsilon|\delta). \tag{3-5}$$

On the other hand, by (3-2-3), we have

$$t_5/t_9 < 20\delta \quad \text{and} \quad \delta^2 R < t_5. \tag{3-6}$$

Inequalities (3-5), (3-6), and Toponogov's comparison theorem (see [2, Chapter 2]) imply

$$\operatorname{ang}\left(\frac{Dl_5}{dt}(0), \frac{Dl_6}{dt}(0)\right) > \pi/2 - \tau(\delta) - \tau(\nu|\delta) - \tau(\varepsilon|\delta). \tag{3-7}$$

Next, let $l_{10} : [0, t_{10}] \to M$ be a minimal geodesic satisfying $l_5(0) = l_{10}(0)$ and $d(l'_6(-t_6/\delta), l_{10}(t_{10})) < \varepsilon$. Then, since

$$|d(l_6(t_6), l_{10}(t_{10})) - (t_6 + t_{10})| < \tau(\nu) + \tau(\varepsilon),$$

it follows that

$$\left|\operatorname{ang}\left(\frac{Dl_6}{dt}(0), \frac{Dl_{10}}{dt}(0)\right) - \pi\right| < \tau(\nu|\delta) + \tau(\varepsilon|\delta). \tag{3-8}$$

On the other hand, by the method used to show (3-7), we can prove

$$\operatorname{ang}\left(\frac{Dl_5}{dt}(0), \frac{Dl_{10}}{dt}(0)\right) > \pi/2 - \tau(\delta) - \tau(\nu|\delta) - \tau(\varepsilon|\delta). \tag{3-9}$$

The lemma follows immediately from inequalities (3-7), (3-8), (3-9).

Remark that to prove Lemma 2-1 we may assume $\delta = \delta'$. When $t_2, t'_2 < \delta R$, clearly we can take $\delta = \delta'$, and when $t_2, t'_2 \geqslant \delta R$, Assertion 3-10 implies that we can replace l_2, l'_2 by $l_2|_{[0, \delta R]}, l'_2|_{[0, \delta R]}$.

Assertion 3-10. $d(l_2(\delta R), l'_2(\delta R)) < \tau(\nu|\delta, \delta') + \tau(\varepsilon|\delta, \delta')$.

Proof. Take minimal geodesics $l'_{11} : [0, R] \to N$ and $l_{11} : [0, t_{11}] \to M$ satisfying $l'_2(0) = l'_{11}(0)$, $d(l_2(\delta R), l'_{11}(\delta R)) < 2\nu + 2\varepsilon$, $l_2(0) = l_{11}(0)$, and $d(l_{11}(t_{11}), l'_{11}(t'_2)) < \varepsilon$. Let $l_{12} : [0, t_{12}] \to M$ denote the minimal geodesics joining $l_2(\delta R)$ to $l_{11}(t_{11})$. Then, since $|t_{12} + \delta R - t_{11}| < \tau(\nu) + \tau(\varepsilon)$ and since l_{11} is minimal, it follows that

$$\operatorname{ang}\left(\frac{Dl_2}{dt}(\delta R), \frac{Dl_{12}}{dt}(0)\right) < \tau(\nu|\delta, \delta') + \tau(\varepsilon|\delta, \delta').$$

Hence we have

$$d(l_2(t_2), l_{11}(t_2)) < \tau(\nu|\delta, \delta') + \tau(\varepsilon|\delta, \delta').$$

On the other hand, by assumption, we have

$$d\big(l_2(t_2), l_2'(t_2')\big) < \nu, \qquad d\big(l_{11}(t_{11}), l_{11}'(t_2')\big) < \varepsilon.$$

Then, we conclude

$$d\big(l_2'(t_2'), l_{11}'(t_2')\big) < \tau(\nu \,|\, \delta, \delta') + \tau(\varepsilon \,|\, \delta, \delta').$$

Therefore, applying Toponogov's comparison theorem to N, we obtain

$$d\big(l_2'(\delta R), l_{11}'(\delta R)\big) < \tau(\nu \,|\, \delta, \delta') + \tau(\varepsilon \,|\, \delta, \delta').$$

The assertion follows from the above inequality and the fact $d(l_2(\delta R), l_{11}'(\delta R)) < \varepsilon$.

Therefore, in the rest of this section, we shall assume $\delta = \delta'$. Take a minimal geodesic $l_{13}: [0, t_{13}] \to M$ joining $l_1(t_1)$ to $l_2(t_2)$. Let $\tilde{l}_i: [0, t_i] \to BT_R(l_1(0), M)$ ($i = 1, 2, 13$) denote the lifts to l_i such that $\tilde{l}_i(0) = 0$ ($i = 1, 2$) and $\tilde{l}_{13}(0) = \tilde{l}_1(t_1)$.

Assertion 3-11. *We have* $d(\tilde{l}_{13}(t_{13}), \tilde{l}_2(t_2)) < (\tau(\delta) + \tau(\nu \,|\, \delta) + \tau(\varepsilon \,|\, \delta)) \cdot \delta$.

Proof. Put $\iota = d(\tilde{l}_{13}(t_{13}), \tilde{l}_2(t_2))$. We may assume $\delta^2 R < \iota$. Take another lift $\hat{l}_2$ of l_2 satisfying $\hat{l}_2(t_2) = \tilde{l}_{13}(t_{13})$. Let $\tilde{l}_i: [0, t_i] \to BT_R(l_1(0), M)$ ($i = 14, 15$) denote the minimal geodesics joining $\tilde{l}_2(t_2)$ to $\tilde{l}_{13}(t_{13})$ and $\tilde{l}_1(0)$ to $\hat{l}_2(0)$ respectively. Then Lemma 3-1 implies

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_2}{dt}(0), \frac{D\tilde{l}_{15}}{dt}(0)\right) - \pi/2\right| < \tau(\delta) + \tau(\nu \,|\, \delta) + \tau(\varepsilon \,|\, \delta),$$

$$\left|\operatorname{ang}\left(\frac{D\hat{l}_2}{dt}(0), \frac{D\tilde{l}_{15}}{dt}(t_{15})\right) - \pi/2\right| < \tau(\delta) + \tau(\nu \,|\, \delta) + \tau(\varepsilon \,|\, \delta),$$

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_2}{dt}(t_2), \frac{D\tilde{l}_{14}}{dt}(0)\right) - \pi/2\right| < \tau(\delta) + \tau(\nu \,|\, \delta) + \tau(\varepsilon \,|\, \delta),$$

$$\left|\operatorname{ang}\left(\frac{D\hat{l}_2}{dt}(t_2), \frac{D\tilde{l}_{14}}{dt}(t_{14})\right) - \pi/2\right| < \tau(\delta) + \tau(\nu \,|\, \delta) + \tau(\varepsilon \,|\, \delta),$$

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_1}{dt}(0), \frac{D\tilde{l}_{15}}{dt}(0)\right) - \pi/2\right| < \tau(\delta) + \tau(\nu \,|\, \delta) + \tau(\varepsilon \,|\, \delta),$$

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_{13}}{dt}(t_{13}), \frac{D\tilde{l}_{14}}{dt}(t_{14})\right) - \pi/2\right| < \tau(\delta) + \tau(\nu \,|\, \delta) + \tau(\varepsilon \,|\, \delta),$$

Hence, a standard argument using Toponogov's comparison theorem implies

$$\begin{aligned} &d(\tilde{l}_{13}(0), \tilde{l}_1(t_1)) \\ &\quad > \iota\{1 - \tau(\delta) - \tau(\nu \,|\, \delta) - \tau(\varepsilon \,|\, \delta)\} - \delta\{\tau(\delta) - \tau(\nu \,|\, \delta) - \tau(\varepsilon \,|\, \delta)\}. \end{aligned}$$

But $\tilde{l}_{13}(0) = \tilde{l}_1(t_1)$. The assertion follows immediately.

Now we are in the position to complete the proof of Lemma 2-1. Assertion 3-11 implies

$$|d(\tilde{l}_1(t_1), \tilde{l}_2(t_2)) - d(l_1'(t_1), l_2'(t_2))| < 2\varepsilon + \delta\{\tau(\delta) + \tau(\nu|\delta) + \tau(\varepsilon|\delta)\}.$$

On the other hand, we have

$$|t_i - t_i'| < 2\nu \quad \text{and} \quad \delta R/10 < t_i < \delta R \qquad (i = 1, 2).$$

Hence, Toponogov's comparison theorem implies

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_1}{dt}(0), \frac{D\tilde{l}_2}{dt}(0)\right) - \operatorname{ang}\left(\frac{Dl_1'}{dt}(0), \frac{Dl_2'}{dt}(0)\right)\right| < \tau(\delta) + \tau(\nu|\delta) + \tau(\varepsilon|\delta),$$

as required.

4. f is an "almost Riemannian submersion"

In this section we shall verify (0-1-13). First we shall prove the following:

Lemma 4-1. $|df| < 1 + \tau(\sigma) + \tau(\varepsilon|\sigma)$.

Proof. Since the second fundamental form of $f_N(N)$ is smaller than $\tau(\sigma)$, the norm of the restriction of the exponential map to $B_{4\varepsilon K^{1/2}}(Nf_N(N))$ is greater than $1 - \tau(\sigma) - \tau(\varepsilon|\sigma)$ (for details, see the proof of [8, Lemma 7.2]). Therefore Lemma 4-1 follows from Lemma 2-3 and the definition of f.

Let $p \in M$, $q = f(p)$. Put $k =$ (the dimension of N). We introduce a new small positive constant θ and assume $\sigma < \theta$. Take points $z_1', z_2', \cdots, z_k'$ of N such that $d(q, z_i') = \theta R$ and that the set of vectors $\operatorname{grad}_q(d(z_1', \cdot)), \cdots, \operatorname{grad}_q(d(z_k', \cdot))$ is an orthonormal base of $T_q(N)$. Let z_i be a point of M such that $d(z_i, z_i') < \varepsilon$. For $x \in B_{\theta^2 R}(p, M)$, put

$$g_i(x) = \int_{y \in B_\varepsilon(z_i, M)} d(x, y)\, dy/\mathbf{Vol}(B_\varepsilon(z_i, M)),$$

and let $\Pi_1(x)$ denote the linear subspace of $T_x(M)$ spanned by $\operatorname{grad}_x(g_1), \cdots, \operatorname{grad}_x(g_k)$, and $\Pi_2(x)$ the orthonormal complement of $\Pi_1(x)$. $P_i : T_x(M) \to \Pi_i(x)$ denotes the orthonormal projections.

Lemma 4-2. *For each $\xi \in \Pi_1(x)$ satisfying $|\xi| = 1$, we have*

$$||df(\xi)| - |\xi|| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

Proof. By Lemmas 2-4, 2-9, and the definitions of f_M, f_N and g_i, we can prove

$$\left|df_M(\operatorname{grad}_x(g_i)) - df_N\left(\operatorname{grad}_{f(x)}\left(d(z_i', \cdot)\right)\right)\right| < (\tau(\sigma) + \tau(\varepsilon|\sigma)) \cdot K^{1/2}.$$

Therefore, by the definition of f, we have

$$|df(\mathrm{grad}_x(g_i)) - \mathrm{grad}_{f(x)}(d(z_i', \cdot))| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

It follows that

$$||df(\mathrm{grad}_x(g_i))| - 1| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

This inequality, combined with Lemma 4-1, implies Lemma 4-2.

The following lemma is a direct consequence of Lemmas 4-1 and 4-2 and the fact $\mathbf{dim}\,\Pi_2(p) = \mathbf{dim}\,N$.

Lemma 4-3. *Let $x \in B_{\theta^2 R}(p, M)$. Then for each $\xi \in T_x(M)$ tangent to the fiber, we have*

$$|P_1(\xi)|/|\xi| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

Now, (0-1-3) follows immediately from Lemmas 4-1, 4-2, and 4-3.

In the rest of this section, we shall prove several lemmas required in the argument of the next section.

Lemma 4-4. *Let $x \in B_{\theta^x R}(p, M)$ and let $\xi \in \Pi_1(x)$ be a vector with $|\xi| = 1$. Then we have*

$$|d(x, \mathbf{exp}_x(s\xi)) - s| < \tau(\sigma) - \tau(\varepsilon|\sigma)$$

and

$$|d(f(x), f(\mathbf{exp}_x(s\xi))) - s| < \tau(\sigma) - \tau(\varepsilon|\sigma)$$

for each s smaller than R.

Proof. Put $\xi' = df(\xi)$, and $l'(t) = \mathbf{exp}(t\xi'/|\xi'|)$. Lemma 4-2 implies $||\xi'| - 1| < \tau(\sigma) + \tau(\varepsilon|\sigma)$. Let $l: [0, R] \to M$ be a minimal geodesic satisfying $d(l(R), l'(R)) < 4\varepsilon + R(|\xi'| - 1)$. Put $\eta = (Dl/dt)(0)$. By Lemma 2-3 and the definition of f, we have

$$|df(\eta) - \xi'| < \tau(\sigma) + \tau(\varepsilon|\sigma). \tag{4-5}$$

Hence we have $||df(\eta)| - |\eta||/|\eta| < \tau(\sigma) + \tau(\varepsilon|\sigma)$, Therefore, Lemmas 4-1, 4-2 imply

$$|P_1(\eta) - \eta| < \tau(\sigma) + \tau(\varepsilon|\sigma). \tag{4-6}$$

Inequalities (4-5), (4-6), combined with the facts $\xi \in \Pi_1(x)$, $df(\xi) = \xi'$, and Lemmas 4-1, 4-2, imply $|\eta - \xi| < \tau(\sigma) + \tau(\varepsilon|\sigma)$. Furthermore, by the definition of η, we have

$$|d(f(x), f(\mathbf{exp}_x(s\eta))) - s| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

The lemma follows immediately from these facts.

Lemma 4-7. *Let $x \in B_{\theta^2 R}(p, M)$, and $\xi_1, \xi_2 \in \Pi_1(x)$ be vectors such that $|\xi_1| = |\xi_2| < \sigma R$. Then we have*

$$|d(\mathbf{exp}(\xi_1), \mathbf{exp}(\xi_2)) - 2 \cdot |\xi_1| \cdot \mathbf{sin}(\mathrm{ang}(\xi_1, \xi_2)/2)| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

Proof. By Lemma 4-4, we have

$$|d(q, f(\mathbf{exp}(\xi_i))) - |\xi_i|| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

On the other hand, Lemmas 4-1 and 4-2 imply

$$|\mathrm{ang}(df(\xi_1), df(\xi_2)) - \mathrm{ang}(\xi_1, \xi_2)| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

Hence, applying Toponogov's comparison theorem to N, we obtain the lemma.

Lemma 4-8. *Let $x \in B_{\theta^2 R}(p, M)$ and $\xi \in \Pi_2(x)$ be a vector with $|\xi| = 1$. Then we have*

$$d(f(\mathbf{exp}(s\xi)), f(x)) < (\tau(\sigma) + \tau(\theta) + \tau(\varepsilon|\sigma, \theta)) \cdot s$$

for each positive number s smaller than $\theta^2 R$.

Proof. Put $l_{16}(t) = \mathbf{exp}(t\xi)$. Since $\xi \in \Pi_2(x)$, we have

$$\mathrm{ang}(\xi, \mathrm{grad}_x(g_i)) = \pi/2. \tag{4-9}$$

Lemma 4-8 follows immediately from Lemmas 4-1, 4-2, 4-3, and the following:

Assertion 4-10. *For each $t < s$, we have*

$$\left|\mathrm{ang}\left(\frac{Dl_{16}}{dt}(t), \mathrm{grad}_{l_{16}(t)}(g_i)\right) - \pi/2\right| < \tau(\varepsilon|\theta) + \tau(\theta).$$

Proof. Let $l_k: [0, t_k] \to M$ $(k = 17, 18)$ be minimal geodesics joining x and $l_{16}(t)$ to z_i respectively. By the definition of g_i, we can take l_{17} and l_{18} so that they satisfy

$$\mathrm{ang}\left(\frac{Dl_{17}}{dt}(0), -\mathrm{grad}_x(g_i)\right) < \tau(\varepsilon|\theta), \tag{4-11}$$

$$\mathrm{ang}\left(\frac{Dl_{18}}{dt}(0), -\mathrm{grad}_{l_{16}(t)}(g_i)\right) < \tau(\varepsilon|\theta). \tag{4-12}$$

Let $\tilde{l}_j$ $(j = 16, 17, 18)$ denote the lifts of l_j $(j = 16, 17, 18)$ to $B_R(x, M)$ satisfying $\tilde{l}_{16}(0) = \tilde{l}_{17}(0) = 0$ and $\tilde{l}_{18}(0) = \tilde{l}_{16}(t)$, and let $\tilde{l}_{19}: [0, t_{19}] \to B_R(x, M)$ denote the minimal geodesic joining $\tilde{l}_{17}(t_{17})$ to $\tilde{l}_{18}(t_{18})$. Put $l_{19} = \mathbf{exp}_x \tilde{l}_{19}$. Then Lemma 3-1 implies that one of the following holds:

$$t_{19} < \theta^2 R, \tag{4-13-1}$$

(4-13-2)
$$\left|\mathrm{ang}\left(\frac{Dl_{17}}{dt}(t_{17}), \frac{Dl_{19}}{dt}(0)\right) - \pi/2\right| < \tau(\theta) + \tau(\varepsilon|\theta),$$
$$\left|\mathrm{ang}\left(\frac{Dl_{18}}{dt}(t_{18}), \frac{Dl_{19}}{dt}(t_{19})\right) - \pi/2\right| < \tau(\theta) + \tau(\varepsilon|\theta).$$

If (4-13-2) holds, then applying Toponogov's comparison theorem to $B_R(x, M)$, we obtain

$$t > (1 - \tau(\varepsilon|\theta) - \tau(\theta)) \cdot t_{19}.$$

Then, in each case, we have $d(\tilde{l}_{17}(t_{17}), \tilde{l}_{18}(t_{18})) = t_{19} < 2\theta^2 R$. Therefore, by a standard argument using Toponogov's comparison theorem, we can prove

(4-14)
$$\left| \operatorname{ang}\left(\frac{D\tilde{l}_{16}}{dt}(0), \frac{D\tilde{l}_{17}}{dt}(0) \right) - \operatorname{ang}\left(\frac{D\tilde{l}_{16}}{dt}(t), \frac{D\tilde{l}_{18}}{dt}(0) \right) \right| < \tau(\theta) + \tau(\varepsilon | \delta).$$

Assertion 4-10 follows immediately from (4-9), (4-11), (4-12), and (4-14).

5. The fiber in an infranilmanifold

In this section we shall verify (0-1-2). The following is a direct consequence of Lemma 2-9.

Lemma 5-1. *The diameter of the fiber, $f^{-1}(q)$, is smaller than $\tau(\varepsilon)$.*

If we can obtain an estimate of the second fundamental form of $f^{-1}(q)$, Lemma 5-1 combined with [6, 1.4] would imply (0-1-2). But as was remarked at §1, the map f is only of C^1-class and not necessarily of C^2-class. Hence, it is impossible to estimate the second fundamental form. Then, instead, we shall modify the proof of [6, 1.4] in order to verify (0-1-3). The detailed proof of [6, 1.4] is presented in [1]. Therefore, in the rest of this section, we shall follow [1], mentioning the required modifications.

We introduce a new positive constant ρ smaller than $\theta^2 R$. Let π_ρ denote the local fundamental pseudogroup introduced in [6, 5.6] or [1, 2.2.6] (in [1] the terminology, local fundamental pseudogroup, is not introduced, but the notation π_ρ is defined there). Here we take p as the base point. Following [1, 2.2.3], we let $*$ denote the Gromov's product on π_ρ. For a vector space V, the symbol $A(V)$ denotes the group of all affine transformations of V. Let $m : \pi_\rho \to A(T_p(M))$ denote the affine holonomy map introduced in [1, 2.3], r its rotation part, and t its translation part. The following lemma is proved in [1, 2.3.1].

Lemma 5-2. *For $\alpha, \beta \in \pi_\rho$, we have*
$$d(r(\beta) \circ r(\alpha), r(\beta * \alpha)) \leqslant |t(\alpha)| \cdot |t(\beta)|,$$
$$|t(m(\beta) \circ m(\alpha))| - |t(\beta * \alpha)| \leqslant |t(\alpha)| |t(\beta)| (|t(\alpha) + t(\beta)|).$$

Next we shall prove the following:

Lemma 5-3. *For each $\alpha \in \pi_\rho$, we have*
$$|P_1 \circ r(\alpha) \circ P_1 - P_1| < \tau(\theta) + \tau(\sigma | \theta) + \tau(\rho | \theta) + \tau(\varepsilon | \sigma, \theta).$$

Proof of Lemma 5-3. Put $s =$ (the length of α). Let ξ be an arbitrary element of $\Pi_1(p)$ satisfying $|\xi| = \theta R$. First we shall prove

(5-4)
$$d(\mathbf{exp}(\xi), \mathbf{exp}(r(\alpha)(\xi))) < \tau(\rho | \theta).$$

In fact, let $\tilde{\xi} \in T_0(BT_R(p, M))$ be a vector satisfying $(d(\mathbf{exp}_p))(\tilde{\xi}) = \xi$, let a curve $\tilde{\alpha} : [0, s] \to BT_R(p, M)$ denote the lift of α satisfying $\tilde{\alpha}(0) = 0$, and let $\hat{\xi} \in T_{\tilde{\alpha}(s)}(BT_R(p, M))$ be a vector satisfying $d(\mathbf{exp}_p)(\hat{\xi}) = r(\xi)$. By the definition of r, the vector $\hat{\xi}$ is a parallel translation of $\tilde{\xi}$ along $\tilde{\alpha}$. Let $\tilde{\xi}(t) \in T_{\tilde{\alpha}(t)}(BT_R(p, M))$ denote the parallel translation of $\tilde{\xi}$ along $\tilde{\alpha}|_{[0,t]}$. Set $J_{t_0}(u) = D/dt|_{t=t_0} \mathbf{exp}_{\tilde{\alpha}(t)}(u \cdot \tilde{\xi}(t))$. Since $J_{t_0}(\cdot)$ is a Jacobi field along the geodesic $u \to \mathbf{exp}_{\tilde{\alpha}(t)}(u \cdot \tilde{\xi}(t_0))$, and since $|J_{t_0}(0)| = 1$, it follows that $|J_{t_0}(1)|$ has an upperbound depending only on n and $|\xi|$. Therefore, $\tilde{\xi}(s) = \hat{\xi}$ implies that

$$d(\mathbf{exp}(\tilde{\xi}), \mathbf{exp}(\hat{\xi})) < \int_0^s |J_t(1)|\, dt \leqslant \tau(\rho \,|\, \theta).$$

Inequality (5-4) follows immediately.

(5-4) and Lemma 4-4 imply

$$\text{(5-5)} \quad |d(p, \mathbf{exp}(r(\alpha)(\xi))) - |r(\alpha)(\xi)|| < \tau(\sigma) + \tau(\rho \,|\, \theta) + \tau(\varepsilon \,|\, \sigma).$$

Next we shall prove the following:

Assertion 5-6. *We have*

$$|P_1(r(\alpha)(\xi)) - r(\alpha)(\xi)|/|r(\alpha)(\xi)| < \tau(\theta) + \tau(\sigma \,|\, \theta) + \tau(\rho \,|\, \theta) + \tau(\varepsilon \,|\, \sigma, \theta).$$

Proof. Put $l_{20}(t) = \mathbf{exp}_p(t \cdot r(\alpha)(\xi)/|\xi|)$ and $t_{20} = |\xi|$. Let $l'_{20} : [0, t'_{20}] \to N$ denote the minimal geodesic satisfying $l'_{20}(0) = q$, $d(l_{20}(t_{20}), l'_{20}(t'_{20})) < \varepsilon$, and $l_{21} : [0, t_{21}] \to M$ be a minimal geodesic joining p to $\mathbf{exp}_p(r(\alpha)(\xi))$. Then, by inequality (5-5) and Lemma 2-9, we can apply Lemma 2-1, and obtain

$$\text{(5-7)} \quad \left|\operatorname{ang}\left(\frac{Dl_{21}}{dt}(0),\, r(\alpha)(\xi)\right)\right| < \tau(\theta) + \tau(\sigma \,|\, \theta) + \tau(\rho \,|\, \theta) + \tau(\varepsilon \,|\, \sigma, \theta).$$

On the other hand, by Lemma 2-4 and the definition of f, we have

$$\left|df\left(\frac{Dl_{21}}{dt}(0)\right) - \frac{Dl'_{21}}{dt}(0)\right| < \tau(\sigma) + \tau(\varepsilon \,|\, \sigma).$$

It follows that

$$\left|\left|df\left(\frac{Dl_{21}}{dt}(0)\right)\right| - \left|\frac{Dl_{21}}{dt}(0)\right|\right| \Big/ \left|df\left(\frac{Dl_{21}}{dt}(0)\right)\right| < \tau(\sigma) + \tau(\varepsilon \,|\, \sigma).$$

Therefore, Lemmas 4-1 and 4-2 imply

$$\text{(5-8)} \quad \operatorname{ang}\left(\frac{Dl_{21}}{dt}(0),\, P_1\left(\frac{Dl_{21}}{dt}(0)\right)\right) < \tau(\sigma) + \tau(\varepsilon \,|\, \sigma).$$

Inequalities (5-7) and (5-8) immediately imply the assertion.

Now, Lemma 5-3 follows immediately from inequality (5-5) and Assertion 5-6.

We put $\tau = \tau(\theta) + \tau(\rho\,|\,\theta) + \tau(\sigma\,|\,\theta) + \tau(\varepsilon\,|\,\sigma, \rho, \theta)$. The following lemma corresponds to [1, Proposition 2.1.3].

Lemma 5-9. *For each $\xi \in \Pi_2(p)$ with $|\xi| < \rho$, there exists $\alpha \in \pi_\rho$ satisfying $|\xi - t(\alpha)| < \tau\rho$.*

Proof. By Lemma 4-8, we have

$$d(f(\mathbf{exp}(\xi)), q) < \tau \cdot |\xi|.$$

This formula and Lemma 5-1 imply that

$$d(\mathbf{exp}(\xi), p) < \tau(\varepsilon) + \tau \cdot |\xi|.$$

The lemma follows immediately.

Next we shall prove a lemma corresponding to [1, 2.2.7]. Following the notations there, we define a group $\hat{\pi}_\rho$ as follows. Let $W(\pi_\rho)$ be the free group of words in the elements of π_ρ; let $N_0(\pi_\rho)$ be the set of words $\alpha\beta\gamma^{-1}$ where $\gamma = \alpha * \beta$; let $N(\pi_\rho)$ be the smallest normal subgroup in $W(\pi_\rho)$ which contains $N_0(\pi_\rho)$. Put $\hat{\pi}_\rho = W(\pi_\rho)/N(\pi_\rho)$.

Lemma 5-10. *If ρ is smaller than a constant depending only on n and μ, and if σ and ε are smaller than a constant depending only on n and R, then there exists a natural isomorphism $\hat{\Phi} : \hat{\pi}_\rho \to \pi_1(f^{-1}(q))$.*

Proof. Since f is a fiber bundle and since any μ balls in N are contractible, we see that $\pi_1(f^{-1}(q))$ is isomorphic to the image of $\pi_1(B_C(p, M))$ in $\pi_1(B_{C'}(p, M))$, where $\sigma, \varepsilon < \tau(C) < C < C'/2 < C' < \mu$. Using this remark, we can prove Lemma 5-10 by the same method as [1, Proposition 2.2.7].

Using Lemmas 5-2, 5-9, and 5-10, the arguments of [1, Chapters 3 and 4] stand with little change. Then, we obtain the following result which corresponds to [1, 4.6.5].

Lemma 5-11. *We can choose ρ such that the following holds.*

(i) *The natural map $\pi_\rho \to \hat{\pi}_\rho$ is injective and $\hat{\pi}_\rho = \pi_1(f^{-1}(q), p)$.*

(ii) *$\hat{\pi}_\rho$ has a nilpotent, torsion free normal subgroup $\hat{\Gamma}_\rho$ of finite index. We put $\Gamma_\rho = \hat{\Gamma}_\rho \cap \pi_\rho$.*

(iii) *Γ_ρ is generated by m loops $\gamma_1, \cdots, \gamma_m$ such that each element $\gamma \in \Gamma_\rho$ can uniquely be written as a normal word $\gamma = \gamma_1^{l_1} \cdots \gamma_m^{l_m}$; these generators are adapted to the nilpotent structure, i.e.*

$$\gamma_j \cdot \langle \gamma_1, \cdots, \gamma_i \rangle \cdot \gamma_j^{-1} = \langle \gamma_1, \cdots, \gamma_i \rangle \qquad (1 \leqslant i \leqslant j \leqslant m).$$

Here m denotes the dimension of $f^{-1}(q)$.

Furthermore, Corollary 3.4.2 in [1] implies the following.

Lemma 5-12. *If $\alpha \in \Gamma_\rho$, then $|r(\alpha)| < \tau$.*

Next we shall follow the argument of [1, Chapter 5]. By Corollary 5.1.3 of [1], we have the following:

Lemma 5-13. *The structure of nilpotent groups on* $\hat{\Gamma}_\rho = (\mathbb{Z}^n, \cdot)$ *can be extended to* $\mathbb{R}^n$. *Namely there exists a nilpotent Lie group* $G = (\mathbb{R}^n, \cdot)$ *such that* $\hat{\Gamma}_\rho$ *is a lattice of* G.

Following [1, 5.1.4], we shall introduce a left invariant metric on G.

Definition 5-14. Put $X_i = P_2(t(\gamma_i))$, $Y_i = \mathbf{exp}^{-1}(\gamma_i) \in L$. Here L denotes the Lie algebra of G. We introduce a scalar product on L such that the linear map given by $X_i \to Y_i$ is an isometry between $\Pi_2(p)$ and L, and extend this product by left translation to a Riemannian metric on G.

Let $\overline{B}$ be a subset of M containing $B_{2\rho}(p, M)$ and satisfying $\pi_1(\overline{B}) = \pi_1(f^{-1}(q))$. Let B denote the universal covering space of $\overline{B}$, and $\pi: B \to \overline{B}$ the projection. Take a point $\tilde{p}$ in $\pi^{-1}(p)$. By the method of [1, 5.4], we can prove the following two lemmas.

Lemma 5-15. *For each* $\alpha \in \Gamma_\rho$, *we have*

$$|d(\tilde{p}, \alpha(\tilde{p})) - d_G(e, \alpha)| < \tau.$$

Here d_G *is the distance induced from the metric defined in* 5-14, *and* e *denotes the unit element.*

Lemma 5-16. *The absolute value of the sectional curvature of* G *has an upperbound depending only on the dimension.*

Let $f_G: G \to L^2(\Gamma_\rho)$ be the map defined by $x \to (h(d_G(x, \gamma(\tilde{p}))))_{\gamma \in \Gamma_\rho}$, where h is a function satisfying condition (1-3), and as the number r in (1-3) we take a constant depending only on ρ, R, and n. The restriction of f_G to $B_\rho(e, G)$ is an embedding. Let $d_B: B \to L^2(\Gamma_\rho)$ denote the map defined by $x \to (h(d(x, \gamma(\tilde{p}))))_{\gamma \in \Gamma_\rho}$. Now using Lemmas 5-15 and 5-16 we can repeat the argument of §§1, 2 and obtain the following. The symbol C_5 below denotes a constant depending only on ρ, R and, n.

Lemma 5-17. *Let* B' *be the* C_5*-neighborhood of* $\{\gamma(\tilde{p}) \mid \gamma \in \Gamma_{\rho - C_5}\}$. *Then there exists a map* $\Phi: B' \to B_\rho(e, G)$ *such that the following hold*:

(5-18-1) Φ *has maximal rank.*

(5-18-2) *If* $x \in B'$, $\gamma \in \hat{\Gamma}_\rho$, $\gamma(x) \in B'$, *then* $\gamma(\Phi(x)) = \Phi(\gamma(x))$.

(5-18-3) *If* $x \in B'$, $\xi \in T_x(B')$ *satisfy* $d\Phi(gx) = 0$, *then we have*

$$\operatorname{ang}(d\pi(\xi), \Pi_2(x)) < \tau$$

(*see Lemma* 4.3).

Now we are in the position to complete the proof of (0-1-2). Put $\tilde{F} = \pi^{-1}(f^{-1}(q))$. By Lemma 5-1, we may assume $\tilde{F} \subset B'$ replacing ε by a smaller one if necessary. Hence, by Lemma 5-17, we obtain a map $\tilde{F}/\hat{\Gamma}_\rho \to G/\hat{\Gamma}_\rho$. Fact (5-18-3) and Lemma 4-3 imply that this map is a covering map. Hence $\tilde{F}/\hat{\Gamma}_\rho$ is a nilmanifold. On the other hand, $\tilde{F}/\hat{\Gamma}_\rho$ is a finite covering of $f^{-1}(q)$. Therefore $f^{-1}(q)$ is an infranilmanifold. Thus the verification of (0-1-2) is completed.

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